Properties of Context-Free Languages

Lecture 24
Sections 8.1 - 8.2

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Outline

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Non-Context-Free Languages

- Not all languages are context-free.
- How would we prove that a language is not context-free?

The Pumping Lemma for CFLs

Theorem (Pumping Lemma for CFLs)

Let L be an infinite context-free language. Then there exists a positive integer m such that for any $w \in L$ with $|w| \geq m$, there exist $u, v, x, y, z \in \Sigma^*$ such that $w = uvxyz$ and

- $|vxy| \leq m$,
- $|vy| \geq 1$, *and*
- $uv^i xy^i z \in L$ *for all* $i \geq 0$.

A Non-Context-Free Language

Theorem

The language $L = \{a^n b^n c^n \mid n \geq 0\}$ *is not context-free.*

A Non-Context-Free Language

- Suppose that L is context free.
- Let m be the "pumping length" of L.
- Let $w = \mathbf{a}^m\mathbf{b}^m\mathbf{c}^m$.
- Then $w = uvxyz$ for some $u, v, x, y, z \in \Sigma^*$ and $|vxy| \le m$ and $|vy| \ge 1$.
- Then vxy could span all **a**'s, or some **a**'s and some **b**'s, or all **b**'s, etc., but it could not span an equal number of **a**'s, **b**'s, and **c**'s.
- Therefore, $uv^2xy^2z \notin L$, which is a contradiction.
- Therefore, L is not context-free.

Outline

Closure Properties

Theorem

Let L_1 and L_2 be context-free languages. Then $L_1 \cup L_2$ and $L_1 L_2$ are context, but $L_1 \cap L_2$ and $\overline{L_1}$ are not necessarily context-free.

Closure Properties

Proof.

- Let G_1 and G_2 be context-free grammars such that $L_1 = L(G_1)$ and $L_2 = L(G_2)$ and with start symbols S_1 and S_2, respectively.
- Then a grammar for $L_1 \cup L_2$ has productions

$$S \to S_1 \mid S_2$$

  along with all the productions in G_1 and G_2.
- And a grammar for $L_1 L_2$ has production

$$S \to S_1 S_2$$

  along with all the productions in G_1 and G_2.

□

Closure Properties

Proof.

- Now let
$$L_1 = \{a^n b^n c^m \mid n, m \geq 0\}$$
and
$$L_2 = \{a^n b^m c^m \mid n, m \geq 0\}.$$
- We see that $L_1 \cap L_2$ is not context-free by noting that
$$L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\},$$
which is known not to be context-free.

□

Proof.

- If the complement of a context-free language were context-free, then it would follow that

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

would be context-free, but we know that it isn't.

□

Outline

The Membership Problem

The Membership Problem

Given a context-free grammar G and a string w, is $w \in L(G)$?

- Transform G into CNF and apply the CYK algorithm.

The Emptiness Problem

Given a context-free grammar G, is $L(G) = \varnothing$?

- Apply the algorithm to remove all λ-productions, unit productions, and useless productions.
- If S has been removed, then S was useless and $L(G) = \varnothing$.
- Otherwise, $L(G) \neq \varnothing$.

The Finiteness Problem

The Finiteness Problem

Given a context-free grammar G, is $L(G)$ finite?

- Apply the algorithm to remove all λ-productions, unit productions, and useless productions.
- If any sequence of productions is "recursive," that is, $A \stackrel{*}{\Rightarrow} uAv$, then $L(G)$ is infinite.
- Otherwise, $L(G)$ is finite.
- From a dependency graph, we can determine whether any sequence of productions is recursive.

Outline

Assignment

Assignment

- Section 8.1 Exercise 1, 2, 3.
- Section 8.2 Exercise 5, 7, 21, 22, 23.